ON WEAK δ – RIGID RINGS

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ABSTRACT

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This thesis consists of the definition of weak δ – rigid ring, some propositions and determining the equivalence of the weak δ – rigid ring R and the weak $\bar{\delta}$ – rigid ring $\bar{R}$, the extension rings of R.

As a results, let R be a ring and δ be a derivation of R. A ring R is said to be weak δ – rigid provided $a\delta(a) \in nil(R)$ if and only if $a \in nil(R)$. We can show that a ring R is a weak δ – rigid ring if and only if $T_n(R)$, the $n \times n$ upper triangular matrix ring over R is a weak $\bar{\delta}$ – rigid ring, where $\bar{\delta}$ is a derivation of $T_n(R)$ extended by δ. In the case that R is a semicommutative ring and $\delta(nil(R)) \subseteq nil(R)$, it is weak δ – rigid if and only if $R[x]$, the polynomial ring over R is a weak $\bar{\delta}$ – rigid ring, where $\bar{\delta}$ is a derivation of $R[x]$ extended by δ.

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วิทยานิพนธ์ฉบับนี้ประกอบด้วยการศึกษาและการให้นิยามของ $\delta-$ริจิดริงอ่อน เมื่อ δ เป็นเดริเวชันของริง R และแสดงการสมมูลกันระหว่างริง R ที่เป็น $\delta-$ริจิดริงอ่อน และริง $\bar{R}$ ที่เป็น $\bar{\delta}-$ริจิดริงอ่อน ซึ่งเป็นริงส่วนขยายของริง R

ให้ δ เป็นเดริเวชันของริง R นิยามริง R ว่าเป็น $\delta-$ริจิดริงอ่อน ถ้า $a\delta(a)\in nil(R)$ ก็ต่อเมื่อ $a\in nil(R)$ สำหรับสมาชิก a ใด ๆ ใน R ผลการวิจัยปรากฏว่า R เป็น $\delta-$ริจิดริงอ่อนก็ต่อเมื่อริงเมทริกซ์สามเหลี่ยมบนมิติ $n\times n, T_n(R)$ เป็น $\bar{\delta}-$ริจิดริงอ่อนเมื่อ $\bar{\delta}$ เป็นเดริเวชันของ, $T_n(R)$ ซึ่งขยายจาก δ ในกรณีที่ R เป็นริงกึ่งสลับที่และ $\delta(nil(R))\subseteq nil(R)$ จะได้ว่า R เป็น $\delta-$ริจิดริงอ่อน ก็ต่อเมื่อ ริงพหุนาม $R[x]$ เป็น $\bar{\delta}-$ริจิดริงอ่อน เมื่อ $\bar{\delta}$ เป็นเดริเวชันของ $R[x]$ ซึ่งขยายจาก δ

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CONTENTS

CHAPTER 1

INTRODUCTION

According to Krempa (1996 : 289-300), an endomorphism α of a ring R is said to be rigid if $a\alpha(a)=0$, then $a=0$ for $a\in R$ and we call a ring R an $\alpha-$*rigid ring* if there exists a rigid endomorphism α of a ring R. In 2007, Ouyang introduced the *weak* $\alpha-$*rigid rings* which is a generalization of $\alpha-$rigid rings. Let α be a ring endomorphism of R. Then R is said to be a weak $\alpha-$rigid ring provided that $a\alpha(a)\in nil(R)$ if and only if $a\in nil(R)$, for any $a\in R$, where $nil(R)$ is the set of all nilpotent elements of R. Furthermore, he obtained some properties of weak $\alpha-$rigid rings as follows. Suppose that α is a ring endomorphism of R. Then R is an $\alpha-$rigid ring if and only if R is a weak $\alpha-$rigid ring and reduced. Moreover, he derived some conditions on the equivalence of the weak $\alpha-$rigid ring R and the weak $\bar{\alpha}-$rigid ring $T_n(R)$, the $n\times n$ upper triangular matrix ring over R.In addition, Bhat (2006 :591-595) studied a ring R with a derivation δ and called a ring R a $\delta-$*rigid ring* if $a\delta(a)=0$ implies $a=0$ and he established a relation between a $\delta-$rigid ring and a 2-primal ring.He also found a relation between the prime radical of a $\delta-$rigid ring R and that of the Ore extension of R.

For this study, we shall introduce the class of weak $\delta-$rigid rings which is a generalization of the class of $\delta-$rigid rings and discuss some properties of the weak $\delta-$rigid rings. Furthermore, we shall investigate some conditions on the equivalence between the weak $\delta-$rigid ring R and the weak $\bar{\delta}-$rigid ring $\bar{R}$,the extension ring of R, when a $\bar{\delta}$ is the derivation of the extension ring $\bar{R}$ of R by δ. And in the case of R is a semicommutative ring and $\delta(nil(R))\subseteq nil(R)$ we shall determine the equivalence of the weak $\delta-$rigid ring R and the weak $\bar{\delta}-$rigid ring $R[x]$, the polynomial ring over R.

Objectives of the Study

The objectives of this study are :

1) Introducing the notion of weak $\delta-$rigid ring.

2) Investigating the equivalence of the weak $\delta-$rigid ring R and the weak $\bar{\delta}-$rigid ring $\bar{R}$, the extension ring of R.

3) Determining the equivalence of the weak $\delta-$rigid ring R and the weak $\bar{\delta}-$rigid ring $R[x]$, the polynomial ring over R, where R is a semicommutative ring and $\delta(nil(R)) \subseteq nil(R)$.

Scope and Limitation

Throughout this study, all rings are associative with identity. In this work, modifying the idea of Ouyang (2008 : 103-116), we introduce the class of weak $\delta-$rigid rings which is a generalization of the class of $\delta-$rigid rings and we shall show that R is a weak $\delta-$rigid ring if and only if the $n \times n$ upper triangular matrix ring $T_n(R)$ is weak $\bar{\delta}-$rigid. Moreover, in this case the polynomial ring $R[x]$ is a weak $\bar{\bar{\delta}}-$rigid, where $\bar{\delta}$ is a derivation of $T_n(R)$ and $\bar{\bar{\delta}}$ is a derivation of $R[x]$ compatible with δ.

Expected Benefits

For this study, the weak $\delta-$rigid rings will be defined and some properties of weak $\delta-$rigid rings will be found. Moreover, the equivalence of the weak $\delta-$rigid of R and the weak $\bar{\delta}-$rigid of $\bar{R}$, the extension ring of R, will be shown.

CHAPTER 2
REVIEW OF LITERATURE

In this chapter, we discuss and investigate some properties of $\alpha-$rigid and weak $\alpha-$rigid rings to apply to weak $\delta-$rigid rings. Therefore, some definitions and essential theorems will be presented for this study.

Literature Review

Throughout this section, R denotes an associative ring with identity.

Krempa (1996 : 289-300) defined a rigid endomorphism and the $\alpha-$rigid rings as follows.

An endomorphism of a ring R is called rigid if $a\alpha(a)=0$ implies $a=0$ for $a\in R$. A ring R is called an $\alpha-$rigid ring if there exists a rigid endomorphism α of R.

Bhat (2006 : 591-595) defined a $\delta-$rigid ring as follows.

Let R be a ring. Let α be an automorphism of R and δ be a $\alpha-$derivation of R. He said that R is a $\delta-$rigid ring if $a\delta(a)=0$ implies $a=0$ for $a\in R$. He noted that a ring R with identity 1 is not a $\delta-$rigid ring since $1\delta(1)=0$. Moreover, he established a relation between a $\delta-$rigid ring and a 2-primal ring as follows.

Let R be a $\delta-$ rigid ring. Let α be an automorphism of R such that $\alpha(P(R))=P(R)$, and δ be a $\alpha-$derivation of R such that $\delta(P(R))\subseteq P(R)$ .Then R is 2-primal.

Ouyang (2008 : 103-116) introduced the concept of a weak $\alpha-$rigid ring and studied its property. He said a ring R with an endomorphism α is a weak $\alpha-$rigid ring provided $a\alpha(a)\in nil(R)$ if and only if $a\in nil(R)$. It was easy to see that the notion of weak $\alpha-$rigid rings generalizes that of $\alpha-$rigid rings.

Moreover, he studied on some extension of weak $\alpha-$rigid rings as follow.

Let α be an endomorphism of a ring R , $T_n(R)$ denote the $n\times n$ upper triangular matrix ring over R . The endomorphism α of R is extended to the endomorphism $\bar{\alpha}:T_n(R)\to T_n(R)$ defined by $\bar{\alpha}((a_{ij}))=(\alpha(a_{ij}))$. Then the following statements are equivalent:

(1) R is weak $\alpha-$rigid ;

(2) $T_n(R)$ is weak $\bar{\alpha}-$rigid for any positive integer n.

On this direction, we shall introduce and study the notion of weak $\delta-$rigid rings which is a generalization of the $\delta-$rigid rings. We will show that a ring R is a weak $\delta-$rigid ring if and only if $T_n(R)$ is a weak $\bar{\delta}-$rigid ring, where $\bar{\delta}$ is a derivation of $T_n(R)$ extended by δ. In the case of R is a semicommutative ring and $R[x]$ denote the polynomial ring over R, R is a weak $\delta-$rigid ring if and only if $R[x]$ is a weak $\bar{\delta}-$rigid ring, where $\bar{\delta}$ is a derivation of $R[x]$ extended by δ.

Theoretical Background

Definitions and theorems

For basic definitions, theorems and notation we refer to Bhattacharya, Jain, and Nagpaul (1994), Goodearl (2004) and Kasch (1982). Many of them can also be found in other textbooks on modules and rings theory, e.g. Dauns (1994), Gilbert and Nicholson (1941). Here we recall some notations which are of particular interest for the investigations in this study.

Definition 2.1 A *ring* is a non-empty set Rtogether with two binary operations, that we shall denote by $+$ and $\cdot$ and called *addition* and *multiplication* (also called product), respectively, such that, for all $a,b,c\in R$ the following axioms are satisfied:

(1) $(R,+)$ is an additive abelian group.

(2) $(R,\cdot)$ is a multiplicative semigroup.

(3)Multiplication is distributive (on both sides) to addition; that is, for all $a,b,c\in R$,

$$a\cdot(b+c)=a\cdot b+a\cdot c,\quad (a+b)\cdot c=a\cdot c+b\cdot c.$$

(The two distributive laws are respectively called the *left distributive law*and the *right distributive law*.)We shall usually write simply ab instead of $a\cdot b$ for $a,b\in R$.

Definition 2.2 A *commutative ring* is a ring R in which multiplication is commutative ; that is, $ab = ba$ for all $a, b \in R$.

Definition 2.3 A *ring with identity* is a ring R in which the multiplicative semigroup has an identity element; that is, there exists $e \in R$ such that $ae = a = ea$ for all $a \in R$. The element e is called the *identity* of R. Generally, the identity element is denoted by 1.

Throughout, all rings are rings with identityunless otherwise stated.

Definition 2.4 Let $(R,+,\cdot)$ be a ring, and let S be a non-empty subset of R.Then S is called a *subring* if $(S,+,\cdot)$ is itself a ring.

Theorem 2.5 (Bhattachaya, Jain and Nagpaul) A non-empty subset S of a ring R is a subring if and only if for all $a, b \in S$ we have $a - b \in S$ and $ab \in S$.

Definition 2.6 Let $(R,+,\cdot)$ be a ring. The function $f : R \to R$ is called a ring *endomorphism* if for all $a, b \in R$:

(1) $f(a+b) = f(a) + f(b)$.

(2) $f(a \cdot b) = f(a) \cdot f(b)$.

A *ring automorphism* is a bijective ring endomorphim.

Definition 2.7 A nonempty subset I of a ring R is called an *ideal* of R if the following conditions are satisfied for all $x, y \in I$ and $r \in R$:

(1) $x - y \in I$.

(2) $x \cdot r$ and $r \cdot x \in I$.

Definition 2.8 An element a in a ring R is called *nilpotent* if there exists a positive integer n such that $a^n = 0$. Set of all nilpotent elements of R is denoted by $nil(R)$.

Theorem 2.9 (Bhattachaya, Jain and Nagpaul) Let R be a commutative ring. Then the set of all nilpotent elements in R forms an ideal.

Proposition 2.10 Let R be a ring, and S be a subring of R. Then $nil(S) = nil(R) \cap S$.

Proof Let $a \in nil(S)$.Then $a \in S$ and there exists some positive integer n such that $a^n = 0$. Since S is a subring of R, we have $a \in R$ and there exists some positive integer n such that $a^n = 0$. Thus $a \in nil(R)$. Therefore $a \in nil(R) \cap S$.

Conversely, Let $a \in nil(R) \cap S$, we have $a \in nil(R)$ and $a \in S$. Since $a \in nil(R)$, there exists positive integer n such that $a^n = 0$. Hence $a \in nil(S)$. □

Definition 2.11 An element e in a ring R is called *idempotent* if $e = e^2$.

For any idempotent element $e \in R$, we have $1-e$ is also idempotent and $e = e^k$ for all positive integer k.

Definition 2.12 Let R be a ring, and α be an endomorphism of R. α is called a *rigid* endomorphism if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. A ring R is called to be α *-rigid* if there exists a rigid endomorphism α of R.

Example 2.13 Let $R = \mathbb{C}$, the set of complex numbers with the usual addition and multiplication, and $\alpha : \mathbb{C} \to \mathbb{C}$ be the map defined by $\alpha(a+bi) = a - bi$, for any $a, b \in \mathbb{R}$. Then it can be seen that α is a rigid endomorphism of R.

Definition 2.14 Let R be a ring and α be an endomorphism of R. A ring R is said to be *weak* $\alpha - $*rigid* if $a\alpha(a) \in nil(R)$ is equivalent to $a \in nil(R)$.

Definition 2.15 A ring R is called a *reduced ring* if R has no nonzero nilpotent elements. Equivalently, a ring is reduced if it has no non–zero elements with square zero, that is, $x^2 = 0$ implies $x = 0$.

Proposition 2.16 Let α be an endomorphism of a ring R. If R is an $\alpha -$rigid ring, then R is a reduced ring.

Proof Let R be an $\alpha -$rigid ring and $a^2 = 0$ for $a \in R$. Then

$$a\alpha(a)\alpha(a\alpha(a)) = a\alpha(a^2)\alpha^2(a) = 0.$$

Thus $a\alpha(a) = 0$ and so $a = 0$. Therefore R is reduced. □

Proposition 2.17 Every subring of a weak $\alpha-$rigid ring is a weak $\alpha-$rigid ring.

Proof Let R be a weak $\alpha-$rigid ring, and S be a subring of R. Let $a \in S$ be such that $a\alpha(a) \in nil(S)$. Then $a\alpha(a) \in nil(R)$. Hence, $a \in nil(R)$ since R is a weak $\alpha-$rigid ring. Therefore $a \in nil(S)$. Conversely, let $a \in S$ be such that $a \in nil(S)$. Then $a \in nil(R)$. Hence, $a\alpha(a) \in nil(S)$ since R is a weak $\alpha-$rigid ring. Therefore, $a\alpha(a) \in nil(S)$. □

Definition 2.18 A *derivation* on a ring R is any map $\delta : R \to R$ satisfying the usual sum and product rules for derivatives :

$$\delta(a+b) = \delta(a) + \delta(b) \text{ and } \delta(ab) = \delta(a)b + a\delta(b)$$

for all $a, b \in R$ (If R is noncommutative, it is important to keep the order of the term involving a and b consistent in the product rule).

Note that: $\delta(1) = \delta(1 \cdot 1) = \delta(1)1 + 1\delta(1) = 2\delta(1)$ whence we automatically have $\delta(1) = 0$.

Example 2.19 Let F be a field and $R = F[x]$. Then the formal derivative $\frac{d}{dx}$ is a derivation of R.

Definition 2.20 Let α be an automorphism of a ring R .An $\alpha-$derivation of R is any map $\delta : R \to R$ such that $\delta(ab) = \delta(a)\alpha(b) + a\delta(b)$ for all $a, b \in R$.

Example 2.21 Let α be an automorphism of a ring R and $\delta : R \to R$ is any map. Let $\phi : R \to M_2(R)$ be a homomorphism defined by $\phi(r) = \begin{pmatrix} \alpha(r) & 0 \\ \delta(r) & r \end{pmatrix}$, for all $r \in R$. Then δ is a $\alpha-$derivation of R. Let $r_1, r_2 \in R$ such that

$$\phi(r_1 + r_2) = \begin{pmatrix} \alpha(r_1 + r_2) & 0 \\ \delta(r_1 + r_2) & r_1 + r_2 \end{pmatrix},$$

$$\phi(r_1) + \phi(r_2) = \begin{pmatrix} \alpha(r_1) & 0 \\ \delta(r_1) & r_1 \end{pmatrix} + \begin{pmatrix} \alpha(r_2) & 0 \\ \delta(r_2) & r_2 \end{pmatrix} = \begin{pmatrix} \alpha(r_1 + r_2) & 0 \\ \delta(r_1 + r_2) & r_1 + r_2 \end{pmatrix}.$$

Now $\phi(r_1+r_2)=\phi(r_1)+\phi(r_2)$, therefore $\delta(r_1+r_2)=\delta(r_1)+\delta(r_2)$. Let

$$\phi(r_1r_2)=\begin{pmatrix}\alpha(r_1r_2) & 0\\ \delta(r_1r_2) & r_1r_2\end{pmatrix},$$

$$\phi(r_1)\phi(r_2)=\begin{pmatrix}\alpha(r_1) & 0\\ \delta(r_1) & r_1\end{pmatrix}\begin{pmatrix}\alpha(r_2) & 0\\ \delta(r_2) & r_2\end{pmatrix}=\begin{pmatrix}\alpha(r_1)\alpha(r_2) & 0\\ \delta(r_1)\alpha(r_2)+r_1\delta(r_2) & r_1r_2\end{pmatrix}.$$

Now $\phi(r_1r_2)=\phi(r_1)\phi(r_2)$.Therefore, $\delta(r_1r_2)=\delta(r_1)\alpha(r_2)+r_1\delta(r_2)$. Hence δ is a $\alpha-$ derivation of R.

Definition 2.22 Let α be an automorphism of a ring R and δ be a $\alpha-$derivation of R. We said that R is a $\delta-$*rigid* ring if $a\delta(a)=0$ implies $a=0$ for $a\in R$.

Definition 2.23 A ring R is *semicommutative* if it satisfies the following condition: whenever element a,b in R satisfy $ab=0$, then $acb=0$ for each $c\in R$.

A semicommutative ring is also called an IFP ring (insertion of factor property).

Example 2.24 Let $R=\left\{\begin{pmatrix}p & q & r\\ 0 & p & s\\ 0 & 0 & p\end{pmatrix} \mid p,q,r,s\in\mathbb{R}\right\}$ be a subring of $T_3(R)$.

Assume that $A=\begin{pmatrix}a_1 & a_2 & a_3\\ 0 & a_1 & a_4\\ 0 & 0 & a_1\end{pmatrix}$, $B=\begin{pmatrix}b_1 & b_2 & b_3\\ 0 & b_1 & b_4\\ 0 & 0 & b_1\end{pmatrix}$ and $C=\begin{pmatrix}c_1 & c_2 & c_3\\ 0 & c_1 & c_4\\ 0 & 0 & c_1\end{pmatrix}$ for any

$a_i,b_i,c_i\in\mathbb{R}$, $i=1,2,3,4$. Suppose that

$$AB=\begin{pmatrix}a_1b_1 & a_1b_2+a_2b_1 & a_1b_3+a_2b_4+a_3b_1\\ 0 & a_1b_1 & a_1b_4+a_4b_1\\ 0 & 0 & a_1b_1\end{pmatrix}=0.$$

Then we have $a_1b_1=a_1b_2+a_2b_1=a_1b_4+a_4b_1=a_1b_3+a_2b_4+a_3b_1=0$.

For $a_1=0$ and $b_1\neq 0$, we have $a_2=a_3=a_4=0$. Then $A=0$. For $a_1\neq 0$ and $b_1=0$, we have $b_2=b_3=b_4=0$. Then $B=0$. And for $a_1=0$ and $b_1=0$, we have $a_2b_4=0$. Hence $ACB=0$. Therefore, R is semicommutative.

Lemma 2.25 (Ouyang) Let R be a semicommutative ring. Then $nil(R)$ is an ideal of R.

Lemma 2.26 (Ouyang) Let R be a semicommutative ring. If $a_0, a_1, \ldots, a_n \in nil(R)$ then $a_0 + a_1x + \ldots + a_nx^n \in R[x]$ is a nilpotent element.

Proposition 2.27 Let R be a semicommutative ring. If $\sum_{i=0}^{n} a_ix^i \in nil(R[x])$ then $a_i \in nil(R)$ for any $i = 0,1,\ldots,n$.

Proof Let $f(x) = \sum_{i=0}^{n} a_ix^i \in R[x]$. We shall show by induction on n that if $\sum_{i=0}^{n} a_ix^i \in nil(R[x])$ then $a_i \in nil(R)$ for any $i = 0,1,\ldots,n$. If $n = 0$, we have $a_0 \in nil(R[x])$. Then $a_0 \in nil(R)$.

Now suppose that k is a positive integer such that if $\sum_{i=0}^{n} a_ix^i \in nil(R[x])$ then $a_i \in nil(R)$ for any $i = 0,1,2,\ldots,n$ and $n < k$. We shall show that if $\sum_{i=0}^{n} a_ix^i \in nil(R[x])$ then $a_i \in nil(R)$ for any $i = 0,1,2,\ldots,n$ and $n = k$.

Suppose $a_0 + a_1x + \cdots + a_kx^k \in nil(R[x])$. Thus $a_0, a_k \in nil(R)$ and hence $a_0 + a_kx^k \in nil(R[x])$, which implies that $a_1x + \cdots + a_{k-1}x^{k-1} \in nil(R[x])$. By induction hypothesis we have $a_1, a_2, \ldots, a_{k-1} \in nil(R)$. Therefore if $\sum_{i=0}^{n} a_ix^i \in nil(R[x])$ then $a_i \in nil(R)$ for any $i = 0,1,\ldots,n$. □

CHAPTER 3

WEAK$\alpha-$RIGID RINGS AND THEIR EXTENSIONS

Throughout this chapter, all rings are defined to be associative with identity. We shall investigate some properties of a $\alpha-$rigid and weak $\alpha-$rigid ring to apply on a weak $\delta-$rigid ring.

Weak $\alpha-$ rigid Rings

We recall that a ring R with an endomorphism α is called a weak $\alpha-$rigid ring provided $a\alpha(a)\in nil(R)$ if and only if $a\in nil(R)$ for any $a\in R$. It is easy to see that the notion of a weak $\alpha-$rigid ring generalizes that of an $\alpha-$rigid ring. The following example shows that there exists a weak $\alpha-$rigid ring R which is not an $\alpha-$rigid ring.

Example 3.1 Let α be an endomorphism of R and R be an a $\alpha-$rigid ring. Let

$$R_3=\left\{\begin{pmatrix} a & b & c\\ 0 & a & d\\ 0 & 0 & a\end{pmatrix} \mid a,b,c,d\in R\right\}$$

be a subring of $T_3(R)$. The endomorphism α of R is extended to the endomorphism $\bar{\alpha}:R_3\to R_3$ defined by $\bar{\alpha}((a_{ij}))=(\alpha(a_{ij}))$. We show that

(1) R_3 is a weak $\bar{\alpha}-$rigid ring.

(2) R_3 is not $\bar{\alpha}-$rigid.

Proof (1) Suppose that

$$\left(\begin{pmatrix} a & b & c\\ 0 & a & d\\ 0 & 0 & a\end{pmatrix}\bar{\alpha}\begin{pmatrix} a & b & c\\ 0 & a & d\\ 0 & 0 & a\end{pmatrix}\right)=\begin{pmatrix} a\alpha(a) & * & *\\ 0 & a\alpha(a) & *\\ 0 & 0 & a\alpha(a)\end{pmatrix}\in nil(R_3).$$

Then there is a positive integer n such that

$$\begin{pmatrix} a\alpha(a) & * & *\\ 0 & a\alpha(a) & *\\ 0 & 0 & a\alpha(a)\end{pmatrix}^n=\begin{pmatrix} (a\alpha(a))^n & * & *\\ 0 & (a\alpha(a))^n & *\\ 0 & 0 & (a\alpha(a))^n\end{pmatrix}=0$$

Thus $a\alpha(a) \in nil(R)$.Since R is an a $\alpha-$rigid it follows that R is reduced and we have $a\alpha(a)=0$, and so $a=0$. Hence $\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} = \begin{pmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} \in nil(R_3)$

Conversely, assume that $\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \in nil(R_3)$. Then there is a positive integer n such that

$$\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix}^n = \begin{pmatrix} a^n & * & * \\ 0 & a^n & * \\ 0 & 0 & a^n \end{pmatrix} = 0.$$

Thus we get $a \in nil(R)$,and so $a=0$ because R is reduced. So

$$\left(\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \bar{\alpha}\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix}\right) = \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \in nil(R_3).$$

Therefore, R_3 is a weak $\bar{\alpha}-$rigid.

(2) Since R_3 is not reduced, R_3 is not $\bar{\alpha}-$rigid . □

Proposition 3.2 Let α be an endomorphism of a ring R. Then R is $\alpha-$rigid if and only if R is weak $\alpha-$rigid and reduced.

Proof Assume that R is $\alpha-$rigid, then R is reduced. Suppose that $a \in nil(R)$.Then $a=0$, and so $a\alpha(a)=0 \in nil(R)$. If $a\alpha(a) \in nil(R)$ for $a \in R$, then $a\alpha(a)=0$, and so $a=0 \in nil(R)$. Therefore, R is weak $\alpha-$rigid and reduced.

Conversely, suppose that R is weak α - rigid and reduced. Let $a\alpha(a)=0$ for $a \in R$, since R is weak $\alpha-$rigid, $a \in nil(R)$. Thus, $a=0$ because R is reduced. Hence, R is $\alpha-$rigid. □

Extension of Weak $\alpha-$rigid Rings

Let α be an endomorphism of a ring R, and $T_n(R)$ denote the $n\times n$ upper triangular matrix ring over R. Then the endomorphism α of R is extended to the endomorphism $\bar{\alpha}: T_n(R)\to T_n(R)$ defined by

$$\bar{\alpha}\left(\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}\right) = \begin{pmatrix} \alpha(a_{11}) & \alpha(a_{12}) & \cdots & \alpha(a_{1n}) \\ 0 & \alpha(a_{22}) & \cdots & \alpha(a_{2n}) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \alpha(a_{nn}) \end{pmatrix}$$

for any $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \in T_n(R)$. Then we have the following:

Theorem 3.3 Let α be an endomorphism of a ring R. Then the following statements are equivalent:

(1) R is weak $\alpha-$rigid ;

(2) $T_n(R)$ is weak $\bar{\alpha}-$rigid for any positive integer n.

Proof (1)$\Rightarrow$(2) Let $A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \in T_n(R)$ be such that $A\in nil(T_n(R))$.

Then there exists some positive integer m such that

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}^m = \begin{pmatrix} (a_{11})^m & * & \cdots & * \\ 0 & (a_{22})^m & \cdots & * \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & (a_{nn})^m \end{pmatrix} = 0.$$

Thus $a_{ii}\in nil(R); i=1,2,\ldots,n$. Since R is weak $\alpha-$rigid, we get $a_{ii}\alpha(a_{ii})\in nil(R)$. So there exist positive integers t_i such that $(a_{ii}\alpha(a_{ii}))^{t_i}=0; i=1,2,\ldots,n$. Let

$t = Max\{t_i\}; i = 1,2,\ldots,n$, then

$$(A\bar{\alpha}(a))^{m} = \begin{pmatrix} a_{11}\alpha(a_{11}) & * & \cdots & * \\ 0 & a_{22}\alpha(a_{22}) & \cdots & * \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn}\alpha(a_{nn}) \end{pmatrix}^{m} = \begin{pmatrix} 0 & * & \cdots & * \\ 0 & 0 & \cdots & * \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}^{n} = 0.$$

Hence $A\bar{\alpha}(A) \in nil(T_n(R))$.

Now, let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \in T_n(R)$ be such that $A\bar{\alpha}(A) \in nil(T_n(R))$.

Then there is a positive integer m such that

$$\left(\left(A\bar{\alpha}(A)\right)\right)^{m} = \begin{pmatrix} a_{11}\alpha(a_{11}) & * & \cdots & * \\ 0 & a_{22}\alpha(a_{22}) & \cdots & * \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn}\alpha(a_{nn}) \end{pmatrix}^{m} = 0.$$

Thus $a_{ii}\alpha(a_{ii}) \in nil(R); i = 1,2,\ldots,n$. Hence, $a_{ii} \in nil(R)$ since R is weak α-rigid, and so there are some positive integers t_i such that $(a_{ii})^{t_i} = 0; i = 1,2,\ldots,n$. Let $t = Max\{t_i\}; i = 1,2,\ldots,n$, then

$$(A)^{m} = \begin{pmatrix} (a_{11})^{t} & * & \cdots & * \\ 0 & (a_{22})^{t} & \cdots & * \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & (a_{nn})^{t} \end{pmatrix}^{n} = \begin{pmatrix} 0 & * & \cdots & * \\ 0 & 0 & \cdots & * \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}^{n} = 0.$$

Hence, $A \in nil(T_n(R))$. Therefore, $T_n(R)$ is weak $\bar{\alpha}$-rigid.

(2) $\Rightarrow$ (1) Since $T_n(R)$ is weak $\bar{\alpha}$-rigid for any positive integer n, it follows that $T_1(R) = R$ and $\bar{\alpha} = \alpha$. Hence, R is weak α-rigid. □

Given a ring R, the trivial extension of R by R is the ring $T(R,R) = R \oplus R$ with the usual addition and the multiplication

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2).$$

This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r, m \in R$ and the usual matrix operations are used.

Let α be an endomorphism of a ring R .Then α is extended to the endomorphism $\bar{\alpha}: T(R,R) \to T(R,R)$ defined by $\bar{\alpha}\left(\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}\right) = \begin{pmatrix} \alpha(r) & \alpha(m) \\ 0 & \alpha(r) \end{pmatrix}$ for any $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix} \in T(R,R)$.

Corollary 3.4 Let α be an endomorphism of a ring R. Then the trivial extension $T(R,R)$ of R by R is weak $\bar{\alpha}-$rigid if and only if R is weak $\alpha-$rigid.

Proof Suppose R is weak $\alpha-$rigid, then $T_n(R)$ is weak $\bar{\alpha}-$rigid. Since $T(R,R)$ is isomorphic to subring $\left\{\begin{pmatrix} r & m \\ 0 & r \end{pmatrix} \mid r, m \in R\right\}$ of a ring $T_n(R)$. So $T(R,R)$ is weak $\bar{\alpha}-$rigid.

Conversely, assume that $T(R,R)$ is weak $\bar{\alpha}-$rigid. Let $a \in nil(R)$. Then there existsa positive integer n such that $a^n = 0$. Let $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in T(R,R)$, then $A \in nil(T(R,R))$ because $A^n = \begin{pmatrix} a^n & 0 \\ 0 & a^n \end{pmatrix} = 0$. Since $T(R,R)$ is weak $\bar{\alpha}-$rigid, we have $A\bar{\alpha}(A) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} \alpha(a) & 0 \\ 0 & \alpha(a) \end{pmatrix} = \begin{pmatrix} a\alpha(a) & 0 \\ 0 & a\alpha(a) \end{pmatrix} \in nil(T(R,R))$. Then there existsa positive integer t such that $(A\bar{\alpha}(A))^t = \begin{pmatrix} a\alpha(a)^t & 0 \\ 0 & a\alpha(a)^t \end{pmatrix} = 0$. Hence, $a\alpha(a) \in nil(R)$. Therefore, R is weak $\alpha-$rigid. □

CHAPTER 4

ON WEAK $\delta -$RIGID RINGS

Throughout this chapter, all rings are associative with identity. In this chapter, we defined weak $\delta -$rigid rings which is a generalization of $\delta -$rigid rings. By definition, we will show that a ring R is a weak $\delta -$rigid ring if and only if $T_n(R)$ is a weak $\bar{\delta} -$rigid ring, where $\bar{\delta}$ is a derivation of $T_n(R)$ extended by δ. In the case that R is a semicommutative ring, it is weak $\delta -$rigid if and only if $R[x]$ is a weak $\bar{\delta} -$rigid ring, where $\bar{\delta}$ is a derivation of $R[x]$ extended by δ .

Weak $\delta -$rigid Rings

In this section we begin with the definition of the weak $\delta -$rigid ring and introduce some properties of weak $\delta -$rigid ring.

Recall that for any semicommutative ring R, $nil(R)$ is an ideal of R and if $a_0, a_1, \ldots, a_n \in nil(R)$ then $a_0 + a_1x + \ldots + a_nx^n \in nil(R[x])$.

Definition 4.1 Let R be a ring and δ be a derivation of R. A ring R is said to be *weak $\delta -$rigid* provided $a\delta(a) \in nil(R)$ if and only if $a \in nil(R)$.

By the definition of weak $\delta -$rigid ring, we have the following proposition and theorem.

Proposition 4.2 Let R be a reduced ring and δ be a derivation of R. Then R is $\delta -$rigid if and only if R is weak $\delta -$rigid.

Proof Assume that a reduced ring R is $\delta -$rigid. Suppose that $a \in nil(R)$ .Since R is a reduced ring, it follows that $a = 0$ .So $a\delta(a) = 0 \in nil(R)$. If $a\delta(a) \in nil(R)$ for $a \in R$, then $a\delta(a) = 0$ .Since R is a $\delta -$rigid, we have $a = 0 \in nil(R)$. Therefore, R is weak $\delta -$rigid.

Conversely, suppose that R is weak $\delta-$rigid. Let $a\delta(a)=0$ for $a\in R$. Then $a\delta(a)\in nil(R)$. Since R is weak $\delta-$rigid, we have $a\in nil(R)$. Thus $a=0$ because R is reduced. Therefore R is $\delta-$rigid. □

For any subring S of a ring R, let δ be a derivation of S. It is easy to see that δ can be extended to the derivation $\bar{\delta}$ of R such that $\bar{\delta}|_S=\delta$.

Theorem 4.3 Every subring of a weak $\delta-$rigid ring is a weak $\delta-$rigid ring.

Proof Let S be a subring of R and R be a weak $\bar{\delta}-$rigid ring. Let $a\in S$ be such that $a\delta(a)\in nil(S)$. Then $a\bar{\delta}(a)=a\delta(a)\in nil(R)$. Hence, $a\in nil(R)$ since R is a weak $\bar{\delta}-$rigid ring. Therefore $a\in nil(S)$. Conversely, let $a\in S$ be such that $a\in nil(S)$ so we have $\bar{\delta}(a)=\delta(a)\in S$. Then $a\in nil(R)$. Hence, $a\bar{\delta}(a)\in nil(R)$ since R is a weak $\delta-$rigid ring. Therefore, $a\delta(a)\in nil(S)$. □

Extension of Weak $\delta-$rigid Rings

Let δ be a derivation of a ring R. Let $T_n(R)$ denote the $n\times n$ upper triangular matrix ring over R. Then the derivation δ of R can be extended to the mapping $\bar{\delta}:T_n(R)\to T_n(R)$ defined by

$$\bar{\delta}\left(\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}\right)=\begin{pmatrix} \delta(a_{11}) & \delta(a_{12}) & \cdots & \delta(a_{1n}) \\ 0 & \delta(a_{22}) & \cdots & \delta(a_{2n}) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \delta(a_{nn}) \end{pmatrix}$$

for any $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}\in T_n(R)$, we have $\bar{\delta}$ is a derivation of $T_n(R)$.

Theorem 4.4 Let δ be a derivation of a ring R. Then the following statements are equivalent :

(1) R is weak $\delta-$rigid ;

(2) $T_n(R)$ is weak $\bar{\delta}-$rigid for any positive integer n.

Proof (1)$\Rightarrow$(2) Let $A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}\in T_n(R)$ be such that $A\in nil(T_n(R))$.

Then there exists a positive integer m such that

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}^m=\begin{pmatrix} (a_{11})^m & * & \cdots & * \\ 0 & (a_{22})^m & \cdots & * \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & (a_{nn})^m \end{pmatrix}=0.$$

Thus $a_{ii}\in nil(R)$ for any $i=1,2,\ldots,n$. Since R is weak $\delta-$rigid, we get $a_{ii}\delta(a_{ii})\in nil(R)$ for any $i=1,2,\ldots,n$. Hence, there exists a positive integer t_i such that $(a_{ii}\delta(a_{ii}))^{t_i}=0$ for any $i=1,2,\ldots,n$. Let $t=Max\{t_i\};i=1,2,\ldots,n$, then

$$(A\bar{\delta}(A))^{tn}=\begin{pmatrix} a_{11}\delta(a_{11}) & * & \cdots & * \\ 0 & a_{22}\delta(a_{22}) & \cdots & * \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn}\delta(a_{nn}) \end{pmatrix}^{tn}=\begin{pmatrix} 0 & * & \cdots & * \\ 0 & 0 & \cdots & * \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}^n=0.$$

Hence $A\bar{\delta}(A)\in nil(T_n(R))$.

Now, let $A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}\in T_n(R)$ be such that $A\bar{\delta}(A)\in nil(T_n(R))$.

Then there is a positive integer m such that

$$(A\bar{\delta}(A))^m=\begin{pmatrix} a_{11}\delta(a_{11}) & * & \cdots & * \\ 0 & a_{22}\delta(a_{22}) & \cdots & * \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn}\delta(a_{nn}) \end{pmatrix}^m=0.$$

Thus $a_{ii}\delta(a_{ii}) \in nil(R)$ for any $i = 1,2,\ldots,n$. Hence $a_{ii} \in nil(R)$ for any $i = 1,2,\ldots,n$, since R is weak δ-rigid. It follows that there exists a positive integers t_i such that $(a_{ii})^{t_i} = 0$ for any $i = 1,2,\ldots,n$. Let $t = Max\{t_i\}; i = 1,2,\ldots,n$. Then

$$(A)^{tn} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}^{tn} = \begin{pmatrix} (a_{11})^t & * & \cdots & * \\ 0 & (a_{22})^t & \cdots & * \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & (a_{nn})^t \end{pmatrix}^{n} = \begin{pmatrix} 0 & * & \cdots & * \\ 0 & 0 & \cdots & * \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}^{n} = 0.$$

Hence $A \in nil(T_n(R))$. Therefore, $T_n(R)$ is weak $\overline{\delta}$ − rigid.

(2) $\Rightarrow$ (1) Since $T_n(R)$ is weak $\overline{\delta}$ − rigid for any positive integer n it follows that $T_1(R) = R$ and $\overline{\delta} = \delta$. Hence R is weak δ − rigid. □

Given a ring R, the trivial extension of R by R is the ring $T(R,R) = R \oplus R$ with the usual addition and the multiplication

$$(r_1,m_1)(r_2,m_2) = (r_1r_2, r_1m_2 + m_1r_2).$$

This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r,m \in R$ and the usual matrix operations are used.

Let δ be a derivation of a ring R. Then δ can be extended to the derivation $\overline{\delta} : T(R,R) \to T(R,R)$ defined by $\overline{\delta}\left(\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}\right) = \begin{pmatrix} \delta(r) & \delta(m) \\ 0 & \delta(r) \end{pmatrix}$ for any $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix} \in T(R,R)$.

Corollary 4.5 Let δ be a derivation of a ring R. Then the trivial extension $T(R,R)$ of R by R is weak $\overline{\delta}$ − rigid if and only if R is weak δ − rigid.

Proof Suppose R is weak $\delta-$rigid. Then $T_2(R)$ is weak $\overline{\delta}-$rigid. Since $T(R,R)$ is isomorphic to a subring $\left\{\begin{pmatrix} r & m \\ 0 & r \end{pmatrix} \mid r,m \in R\right\}$ of thering $T_2(R)$, we can see that $T(R,R)$ is weak $\overline{\delta}-$rigid.

Conversely, assume that $T(R,R)$ is weak $\overline{\delta}-$rigid. Let $a \in nil(R)$. Then there existsa positive integer n such that $a^n = 0$. Let $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in T(R,R)$, then $A \in nil(T(R,R))$. Since $T(R,R)$ is weak $\overline{\delta}-$rigid, we have $A\overline{\delta}(A) \in nil(T(R,R))$. Hence, we can find a positive integer t such that $(A\overline{\delta}(A))^t = \begin{pmatrix} a\delta(a) & 0 \\ 0 & a\delta(a) \end{pmatrix}^t =$ $\begin{pmatrix} (a\delta(a))^t & 0 \\ 0 & (a\delta(a))^t \end{pmatrix} = 0$. Thus $a\delta(a) \in nil(R)$. Therefore, R is weak $\delta-$rigid. □

Let δ be a derivation of a ring R. Then the derivation δ of R can be extended to the mapping $\overline{\delta} : R[x] \to R[x]$ defined by

$$\overline{\delta}\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} \delta(a_i) x^i$$

for any $\sum_{i=0}^{n} a_i x^i \in R[x]$.Then $\overline{\delta}$ is a derivation of $R[x]$.

Theorem 4.6 Let R be a semicommutative ring, δ be a derivation of R and $\delta(nil(R)) \subseteq nil(R)$. Then the following statements are equivalent :

(1) R is weak $\delta-$rigid ;

(2) $R[x]$ is weak $\overline{\delta}-$rigid.

Proof (1) $\Rightarrow$ (2) Let $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ be such that $f(x) \in nil(R[x])$. Then $a_i \in nil(R)$ for any $i = 0,1,\ldots,n$. Since R is semicommutative, we have $nil(R)$ is an ideal. Then $a_i\delta(a_j) \in nil(R)$ for any $i = 0,1,\ldots,n$ and $j = 0,1,\ldots,n$. Therefore,

$$f(x)\overline{\delta}(f(x)) = (\sum_{i=0}^{n} a_i x^i)(\sum_{j=0}^{n} \delta(a_j) x^j) = \sum_{k=0}^{2n}\left(\sum_{i+j=k} a_i\delta(a_j)\right)x^k \in nil(R[x]).$$

Conversely, let $f(x)=\sum_{i=0}^{n} a_i x^i \in R[x]$ such that

$$f(x)\overline{\delta}(f(x))=\sum_{k=0}^{2n}\left(\sum_{i+j=k} a_i\delta(a_j)\right)x^k \in nil(R[x]).$$

Then $\sum_{i+j=k} a_i\delta(a_j)\in nil(R)$ for any $k=0,1,\ldots,2n$.

For $k=0$ then $i=j=0$, we have $a_0\delta(a_0)\in nil(R)$. Since R is weak δ-rigid, we have $a_0\in nil(R)$.

For $k=2$ we have $a_0\delta(a_2)+a_1\delta(a_1)+a_2\delta(a_0)\in nil(R)$. Since $nil(R)$ is an ideal and $\delta(nil(R))\subseteq nil(R)$, we have $a_1\delta(a_1)\in nil(R)$. Thus $a_1\in nil(R)$.

Let $l\in\{0,1,2,\ldots,n\}$ and suppose that $a_m\in nil(R)$ for all $m<l$. We will show that $a_m\in nil(R)$ for $m=l$.

For $k=2l$ we have $a_0\delta(a_{2l})+a_1\delta(a_{2l-1})+\cdots+a_l\delta(a_l)+\cdots a_{2l}\delta(a_0)\in nil(R)$. Since $nil(R)$ is an ideal and $\delta(nil(R))\subseteq nil(R)$. By induction hypothesis we have $a_l\delta(a_l)\in nil(R)$. Then $a_l\in nil(R)$. Thus, $a_m\in nil(R)$ for any $m=0,1,2,\ldots,n$. Hence, $f(x)=\sum_{i=0}^{n} a_i x^i\in nil(R[x])$. Therefore, $R[x]$ is weak $\overline{\delta}$ − rigid.

(2) ⇒ (1) Since $R[x]$ is weak $\overline{\delta}$ − rigid and R is a subring of $R[x]$. Then R is weak δ − rigid. □

CHAPTER 5
CONCLUSIONS

Throughout this study, all rings are defined to be associative with identity. For a ring derivation δ of a ring R, we defined a ring R to be a weak $\delta-$rigid ring if $a\delta(a) \in nil(R)$ if and only if $a \in nil(R)$ for any $a \in R$. We determined the equivalence of the weak $\delta-$rigid ring R and the weak $\bar{\delta}-$rigid ring $\bar{R}$, the extension ring of R as following :

1. Let δ be a derivation of a ring R. Then the following statements are equivalent :

   (1) R is weak $\delta-$rigid ;

   (2) $T_n(R)$ is weak $\bar{\delta}-$rigid for any positive integer n.

2. Let δ be a derivation of ring R. Then the trivial extension $T(R,R)$ of R by R is weak $\bar{\delta}-$rigid if and only if R is weak $\delta-$rigid.

3. Let R be a semicommutative ring, δ be a derivation of R and $\delta(nil(R)) \subseteq nil(R)$. Then the following statements are equivalent :

   (1) R is weak $\delta-$rigid ;

   (2) $R[x]$ is weak $\bar{\delta}-$rigid.

For further research, we can investigate some properties of weak $\delta-$rigid ring and find relation between weak $\delta-$rigid ring and other rings.

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APPENDICES

ภาควิชาคณิตศาสตร์
คณะวิทยาศาสตร์
จุฬาลงกรณ์มหาวิทยาลัย

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ในนามของคณะกรรมการจัดการประชุมวิชาการคณิตศาสตร์บริสุทธิ์และประยุกต์ประจำปี 2554 ขอแจ้งให้ท่านทราบว่า อนุกรรมการฝ่ายวิชาการของการประชุม ฯ ได้พิจารณาบทความของท่านในหัวข้อ

On Weak δ-rigid Rings

เรียบร้อยแล้ว และมีความยินดีขอเชิญท่านเข้าร่วมประชุมและเสนอผลงานในหัวข้อดังกล่าว ทั้งนี้ กำหนดการนำเสนอจะแจ้งให้ท่านทราบต่อไป

นอกจากนี้ ท่านสามารถรับทราบข้อมูลต่าง ๆ ของการประชุมครั้งนี้ ได้ทางเว็บไซต์

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APPENDIX II

ON WEAK $\delta-$RIGID RINGS

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On Weak δ-rigid Rings

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Abstract

For a ring derivation $\delta$of a ring R. We defined a ring R to be a weak δ-rigid ring if $a\delta(a) \in nil(R)$ if and only if $a \in nil(R)$ for any $a \in R$. When $\bar{\delta}$is derivation of $T_n(R)$, the $n \times n$ upper triangular matrix ring, extended by δ, we prove that R is weak δ-rigid ring if and only if $T_n(R)$ is weak $\bar{\delta}$-rigid ring. Moreover, if R is a semicommutative ring $\delta(nil(R)) \subseteq nil(R)$ and $\bar{\delta}$is derivation of the polynomial ring $R[x]$ extended by δ, R is weak δ-rigid ring if and only if $R[x]$ is weak $\bar{\delta}$-rigid ring

Mathematics Subject Classification: Rings and modules theory
Keywords: Derivation, Weak δ-rigid, Semicommutative ring.

1 Introduction

Let R be a associative ring with identity, $\delta : R \to R$ is a derivation of a ring R satisfying the usual sum and product rules for derivatives: $\delta(a+b) = \delta(a) + \delta(b)$ and $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in R$. The polynomial ring with an indeterminate x over R is denote by $R[x]$. An element a in a ring R is called nilpotent if there exists a positive integer n such that $a^n = 0$. Set of all nilpotent element of R are denoted by $nil(R)$. The $n \times n$ upper triangular matrix over R denoted by $T_n(R)$. A ring R is called semicommutative ring if for any element $a, b \in R$ satisfy $ab = 0$, then $acb = 0$ for each $c \in R$.

According to [2], for a ring R with a derivation δ, we called a ring R to be δ-rigid ring if $a\delta(a) = 0$ implies $a = 0$. In this paper we defined the weak δ-rigid rings which are generalization of δ-rigid rings. By definition of weak δ-rigid rings we will show that a ring R is a weak δ-rigid ring if and only if $T_n(R)$ is a weak $\bar{\delta}$-rigid ring, when $\bar{\delta}$ is derivation of $T_n(R)$ extended by δ. And in the case of R is a semicommutative ring, R is weak δ-rigid ring if and only if $R[x]$ is weak $\bar{\delta}$-rigid ring, when $\bar{\delta}$ is derivation of $R[x]$ extended by δ .

2 On Weak δ-rigid Rings

In this section we begin with some lemmas and introduced the weak δ-rigid ring.

Lemma 2.1 ([11], Lemma 3.7). *Let R be a semicommutative ring. Then $nil(R)$ is an ideal.*

Lemma 2.2 ([11], Lemma 3.8). *Let R be a semicommutative ring. If $a_0, a_1, \ldots, a_n \in nil(R)$ then $\sum_{i=0}^{n} a_i x^i \in R[x]$ is a nilpotent element.*

Definition 2.3. Let R be a ring and δ be a derivation of R. A ring R is called a weak δ-rigid if $a\delta(a) \in nil(R)$ if and only if $a \in nil(R)$.

By definition of the weak δ-rigid ring, we have following propositions and theorems.

Proposition 2.4. *Let R be a ring, and S be a subring of R. Then $nil(S) = nil(R) \cap S$.*

Proof. Let $a \in nil(S)$, then $a \in S$ and there exists some positive integer n such that $a^n = 0$. Since S is a subring of R, we have $a \in R$ and there exists some positive integer n such that $a^n = 0$. Thus $a \in nil(R)$. Therefore $a \in nil(R) \cap S$.

Conversely, Let $a \in nil(R) \cap S$, we have $a \in nil(R)$ and $a \in S$. Since $a \in nil(R)$, there exists positive integer n such that $a^n = 0$. Hence $a \in nil(S)$. ◻

For any subring S of a ring R, let δ be a derivation of S. It is easy to se that δ can be extended to the derivation $\bar{\delta}$ of R such that $\bar{\delta}|_S = \delta$.

Theorem 2.5. *Every subring of a weak $\bar{\delta}$-rigid ring is a weak δ-rigid ring.*

Proof. Let S be a subring of R and R be a weak $\bar{\delta}$-rigid ring. Let $a \in S$ be such that $a\delta(a) \in nil(S)$. Then $a\bar{\delta}(a) = a\delta(a) \in nil(R)$. Hence $a \in nil(R)$, since R is a weak $\bar{\delta}$-rigid ring. Therefore $a \in nil(S)$. Conversely, let $a \in S$ be such that $a \in nil(S)$,we have $\bar{\delta}(a) = \delta(a) \in S$. Then $a \in nil(R)$. Hence $a\bar{\delta}(a) \in nil(R)$, since R is a weak δ-rigid ring. Therefore $a\delta(a) \in nil(S)$. ◻

Let δ be a derivation of a ring R. Let $T_n(R)$ denote the $n \times n$ upper triangular matrix ring over R. Then the derivation δ of R can be extended to the mapping $\bar{\delta} : T_n(R) \to T_n(R)$ defined by

$$\bar{\delta}\left(\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}\right) = \begin{pmatrix} \delta(a_{11}) & \delta(a_{12}) & \cdots & \delta(a_{1n}) \\ 0 & \delta(a_{22}) & \cdots & \delta(a_{2n}) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \delta(a_{nn}) \end{pmatrix},$$

for any $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \in T_n(R)$, we have $\bar{\delta}$ is a derivation of $T_n(R)$.

Theorem 2.6. *Let δ be a derivation of a ring R. Then the following statement are equivalent :*
(1) R is weak δ-rigid.
(2) $T_n(R)$is weak $\bar{\delta}$-rigid for any positive integer n.

Proof. (1) $\Rightarrow$ (2) Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \in T_n(R)$ be such that $A \in nil(T_n(R))$. Then there exists some positive integer m such that

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}^m = \begin{pmatrix} (a_{11})^m & * & \cdots & * \\ 0 & (a_{22})^m & \cdots & * \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & (a_{nn})^m \end{pmatrix} = 0$$

Thus $a_{ii} \in nil(R)$; $i = 1,2,\ldots,n$. Since R is weak δ-rigid, we get $a_{ii}\delta(a_{ii}) \in nil(R)$; $i = 1,2,\ldots,n$. So there exists positive integer t_i such that $(a_{ii}\delta(a_{ii}))^{t_i} = 0$; $i = 1,2,\ldots,n$. Let $t = Max\{t_i\}$; $i = 1,2,\ldots,n$, then

$$(A\bar{\delta}(A))^{tn} = \begin{pmatrix} a_{11}\delta(a_{11}) & * & \cdots & * \\ 0 & a_{22}\delta(a_{22}) & \cdots & * \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn}\delta(a_{nn}) \end{pmatrix}^{tn} = \begin{pmatrix} 0 & * & \cdots & * \\ 0 & 0 & \cdots & * \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}^{n} = 0.$$

Hence $A\bar{\delta}(A) \in nil(T_n(R))$.

Now, let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \in T_n(R)$ be such that $A\bar{\delta}(A) \in nil(T_n(R))$. Then there is some positive integer m such that

$$(A\bar{\delta}(A))^{m} = \begin{pmatrix} a_{11}\delta(a_{11}) & * & \cdots & * \\ 0 & a_{22}\delta(a_{22}) & \cdots & * \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn}\delta(a_{nn}) \end{pmatrix}^{m} = 0$$

Thus $a_{ii}\delta(a_{ii}) \in nil(R)$; $i = 1,2,\ldots,n$. Hence $a_{ii} \in nil(R)$; $i = 1,2,\ldots,n$ since R is weak δ-rigid. So there exists some positive integer t_i such that $(a_{ii})^{t_i} = 0$; $i = 1,2,\ldots,n$. Let $t = Max\{t_i\}$; $i = 1,2,\ldots,n$, then

$$(A)^{tn} = \begin{pmatrix} a_{11} & * & \cdots & * \\ 0 & a_{22} & \cdots & * \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}^{tn} = \begin{pmatrix} (a_{11})^t & * & \cdots & * \\ 0 & (a_{22})^t & \cdots & * \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & (a_{nn})^t \end{pmatrix}^{n} = \begin{pmatrix} 0 & * & \cdots & * \\ 0 & 0 & \cdots & * \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}^{n} = 0$$

Hence $A \in nil(T_n(R))$. Therefore $T_n(R)$ is weak $\bar{\delta}$-rigid.

(2) $\Rightarrow$ (1) Let $a \in nil(R)$, then $a^n = 0$ for some positive integer n.

Let $A = \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in T_n(R)$, then $A \in nil(T_n(R))$. Since $T_n(R)$ is weak $\bar{\delta}$-rigid,

$$A\bar{\delta}(A) = \begin{pmatrix} a\delta(a) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in nil(T_n(R)).$$

Thus $a\delta(a) \in nil(R)$. Now let $a\delta(a) \in nil(R)$, then

$$\begin{pmatrix} a\delta(a) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \bar{\delta}\left(\begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}\right) = A\bar{\delta}(A) \in nil(T_n(R))$$

Since $T_n(R)$ is weak $\bar{\delta}$-rigid, we have $A = \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in nil(T_n(R))$, and so $a \in nil(R)$.

Hence R is weak δ-rigid. ∎

Given a ring R, the trivial extension of R by R is the ring $T(R,R) = R \oplus R$ with the usual addition and the multiplication

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2).$$

This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r, m \in R$ and the usual matrix operation are used.

Let δ be a derivation of a ring R, then δ is extended to the derivation $\bar{\delta} : T(R,R) \to T(R,R)$ defined by $\bar{\delta}\left(\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}\right) = \begin{pmatrix} \delta(r) & \delta(m) \\ 0 & \delta(r) \end{pmatrix}$ for any $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix} \in T(R,R)$.

Corollary 2.7. *Let δ be a derivation of ring R. Then the trivial extension $T(R,R)$ of R by R is weak $\bar{\delta}$-rigid if and only if R is weak δ-rigid.*

Proof. Suppose R is weak δ-rigid, then $T_2(R)$ is weak $\bar{\delta}$-rigid. Since $T(R,R)$ is isomorphic to subring $\left\{\begin{pmatrix} r & m \\ 0 & r \end{pmatrix} | r, m \in R\right\}$ of a ring $T_2(R)$. So $T(R,R)$ is weak $\bar{\delta}$-rigid.

Conversely, assume that $T(R,R)$ is weak $\bar{\delta}$-rigid. Let $a \in nil(R)$. Then there exist some positive integer n such that $a^n = 0$. Let $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in T(R,R)$, then $A \in nil(T(R,R))$. Since $T(R,R)$ is weak $\bar{\delta}$-rigid, we have $A\bar{\delta}(A) = \begin{pmatrix} a\delta(a) & 0 \\ 0 & a\delta(a) \end{pmatrix} \in nil(T(R,R))$. Then there exist some positive integer t such that $(A\bar{\delta}(A))^t = \begin{pmatrix} (a\delta(a))^t & 0 \\ 0 & (a\delta(a))^t \end{pmatrix} = 0$. Hence $a\delta(a) \in nil(R)$. Therefore R is weak δ-rigid. ∎

Let δ be a derivation of a ring R. Then the derivation δ of R can be extended to the mapping $\bar{\delta} : R[x] \to R[x]$ defined by

$$\bar{\delta}\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} \delta(a_i) x^i$$

For any $\sum_{i=0}^{n} a_i x^i \in R[x]$, such that $\bar{\delta}$ is a derivation of $R[x]$.

Proposition 2.8. *Let R be a semicommutative ring. If $\sum_{i=0}^{n} a_i x^i \in nil(R[x])$ then $a_i \in nil(R)$ for any $i = 0, 1, \ldots, n$.*

Proof. Let $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$. We will show by induction on n that if $\sum_{i=0}^{n} a_i x^i \in nil(R[x])$ then $a_i \in nil(R)$ for any $i = 0, 1, \ldots, n$. If $n = 0$, we have $a_0 \in nil(R[x])$. Then $a_0 \in nil(R)$.

Now suppose that k is a positive integer such that if $\sum_{i=0}^{n} a_i x^i \in nil(R[x])$ then $a_i \in nil(R)$ for $n < k$. We will show that if $\sum_{i=0}^{n} a_i x^i \in nil(R[x])$ then $a_i \in nil(R)$ for $n = k$.

Suppose $a_0 + a_1 x + \cdots + a_k x^k \in nil(R[x])$. Thus $a_0, a_k \in nil(R)$ and hence $a_0 + a_k x^k \in nil(R[x])$, which implies that $a_1 x + \cdots + a_{k-1} x^{k-1} \in nil(R[x])$. By induction hypothesis we have $a_1, a_2, \ldots, a_{k-1} \in nil(R)$. Therefore if $\sum_{i=0}^{n} a_i x^i \in nil(R[x])$ then $a_i \in nil(R)$ for any $i = 0, 1, \ldots, n$. ∎

Theorem 2.9. *Let R be a semicommutative ring, δ be a derivation of R and $\delta(nil(R)) \subseteq nil(R)$. Then the following statement are equivalent :*

(1) R is weak δ-rigid.

(2) $R[x]$ is weak $\bar{\delta}$-rigid.

Proof. (1) ⇒ (2) Let $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ be such that $f(x) \in nil(R[x])$. Then $a_i \in nil(R)$ for any $i = 0, 1, 2, \ldots, n$. Since R is semicommutative, we have $nil(R)$ is an ideal. Then $a_i\delta(a_j) \in nil(R)$ for any i, j. Therefore $f(x)\bar{\delta}(f(x)) = (\sum_{i=0}^{n} a_i x^i)(\sum_{j=0}^{n} \delta(a_j) x^j) = \sum_{k=0}^{2n}\left(\sum_{i+j=k} a_i\delta(a_j)\right) x^k \in nil(R[x])$.

Conversely, let $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ be such that $f(x)\delta(f(x)) = \sum_{k=0}^{2n} \left(\sum_{i+j=k} a_i\delta(a_j) \right) x^k \in nil(R[x])$. Then $\sum_{i+j=k} a_i\delta(a_j) \in nil(R)$ for any $k = 0, 1, 2, \ldots, 2n$.

If $k = 0$ we have $i = j = 0$, thus $a_0\delta(a_0) \in nil(R)$. Since R is weak δ-rigid, we have $a_0 \in nil(R)$.

If $k = 2$ we have $a_0\delta(a_2) + a_1\delta(a_1) + a_2\delta(a_0) \in nil(R)$. Since $nil(R)$ is an ideal and $\delta(nil(R)) \subseteq nil(R)$, we have $a_1\delta(a_1) \in nil(R)$. Thus $a_1 \in nil(R)$.

If $k = 4$ we have $a_0\delta(a_4) + a_1\delta(a_3) + a_2\delta(a_2) + a_3\delta(a_1) + a_4\delta(a_0) \in nil(R)$. Since $nil(R)$ is an ideal and $\delta(nil(R)) \subseteq nil(R)$, we have $a_2\delta(a_2) \in nil(R)$. Thus $a_2 \in nil(R)$.

Continuing this procedure yield that for any $i = 0, 1, \ldots, n$, $a_i \in nil(R)$. Hence $f(x) = \sum_{i=0}^{n} a_i x^i \in nil(R[x])$. Therefore $R[x]$is weak δ-rigid.

(2) ⇒ (1) Since $R[x]$ is weak δ-rigid and R is a subring of $R[x]$. Then R is weak δ-rigid. □

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